

Mean-Field Gauge Interactions in Five Dimensions II. The Orbifold.

Nikos Irges¹, Francesco Knechtli² and Kyoko Yoneyama²

1. Department of Physics

National Technical University of Athens

Zografou Campus, GR-15780 Athens Greece

2. Department of Physics, Bergische Universität Wuppertal

Gaussstr. 20, D-42119 Wuppertal, Germany

*e-mail: irges@mail.ntua.gr, knechtli@physik.uni-wuppertal.de,
yoneyama@physik.uni-wuppertal.de*

Abstract

We study Gauge-Higgs Unification in five dimensions on the lattice by means of the mean-field expansion. We formulate it for the case of an $SU(2)$ pure gauge theory and orbifold boundary conditions along the extra dimension, which explicitly break the gauge symmetry to $U(1)$ on the boundaries. Our main result is that the gauge boson mass computed from the static potential along four-dimensional hyperplanes is nonzero implying spontaneous symmetry breaking. This observation supports earlier data from Monte Carlo simulations [12].

1 Introduction

The phase diagram of five-dimensional gauge theories is surprisingly rich. On an infinite, hypercubic, anisotropic lattice it is parametrized by the two dimensionless parameters $\beta_4 = \beta/\gamma$ and $\beta_5 = \beta\gamma$, where β is the lattice coupling and γ the anisotropy parameter. In part I of this work [1] we explored the phase diagram of a five-dimensional $SU(2)$ gauge theory with fully periodic boundary conditions, using an expansion around a mean-field background [2]. We concentrated on the regime where $\beta \sim O(1)$ and $\gamma < 1$ where a line of second order phase transitions was observed. In the vicinity of this phase transition the system reduces dimensionally to four dimensions and in the continuum limit the physics is consistent with what one would expect from the lightest states with four dimensional quantum numbers. For the fully periodic system this would be a four dimensional $SU(2)$ gauge theory coupled to an adjoint scalar in the confined phase [3].

In this work we extend the construction of [1] by changing the boundary conditions in the fifth dimension from periodic to orbifold. The embedding of the orbifold projection in the geometry introduces boundaries at the "ends" of the fifth dimension (which is now an interval) and its embedding into the gauge group as well known by now [4], alters the field content surviving on the boundaries. In this respect, for $SU(N)$ there are at least two possibilities. One is when the orbifold action is such that the adjoint set of scalars (i.e. the extra-dimensional components of the gauge field) is projected out at the boundaries and one is left there with just a pure $SU(N)$ gauge theory. This construction allows one to carry out an analysis similar to [3] but in a context that is directly generalizable to QCD once fermions are also added. The other possibility is to use the orbifold action to project out some of the gauge fields and some of the scalars. With such a choice it is possible to realize a field content similar to the one of the bosonic sector of the Standard Model. At a first stage we consider an $SU(2)$ bulk gauge group. The orbifold then leaves a $U(1)$ theory coupled to a complex scalar on the boundaries. This setup could serve as the simplest prototype of the Higgs mechanism.

The idea that the Standard Model Higgs particle may be the remnant of an extra dimensional gauge field is not new [5]. Also, investigations of five-dimensional gauge theories with a lattice regularization have both an analytical and a Monte Carlo past. The analytical work has been concentrated around the question of the existence or not of a layered phase [6] and around the existence or not of an ultraviolet fixed point [7]. Monte Carlo simulations have been mostly looking for the layered phase [8], for first order bulk [9] (the pioneering work in this direction) or second order [10] (finite temperature and bulk) phase transitions and dimensional reduction via localization [11]. All of these lattice investigations have been carried out on fully periodic lattices. Recently there has been interest also in lattices with orbifold boundary conditions [12], [13]. Our motivation

to look more carefully at the phase diagram of five-dimensional orbifold gauge theories stems from the fact that Coleman-Weinberg computations [14], continuum perturbation theory at one loop [15] and exploratory lattice Monte Carlo simulations [12] indicate that once a Higgs mass is generated by quantum effects, it seems to remain finite, despite the non-renormalizable nature of the higher dimensional theory.

A first attempt to probe analytically the regime away from the perturbative point, in order to see if there is a dynamical mechanism of spontaneous symmetry breaking (SSB) triggered by the gauge field (without the presence of fermions) was made in [16]. The idea there was to use the $SU(N)$ Symanzik lattice effective action [17] (a is the lattice spacing and N_5 the number of lattice points in the fifth dimension)

$$-\mathcal{L}_{\text{Sym}} = \frac{\beta}{4Na} \text{tr}\{F \cdot F\} + \sum_{p_i} c^{(p_i)}(N_5, \beta) a^{p_i-4} \mathcal{O}^{(p_i)} + \dots \quad (1.1)$$

which is determined by a finite number of dimensionless coefficient functions $c^{(p_i)}(N_5, \beta)$ on an infinite spatial isotropic lattice, provided that one can consistently truncate the expansion. The ansatz in [16] was to truncate the expansion after the first two higher dimensional operators: one of order a corresponding to the lowest order dimension five boundary counterterm parametrized by the coefficient function $c^{(5)}(N_5, \beta)$ and one of order a^2 corresponding to the dimension six bulk operator $\text{tr}(DF)^2$ parametrized by the coefficient function $c^{(6)}(N_5, \beta)$. To extract information about SSB in this setup, one assumes a vacuum expectation value (vev) v for one of the A_5 components of the five-dimensional gauge field A_M and uses the truncated expansion expanded around this vev to compute a Coleman-Weinberg type potential $V(\alpha)$ for the dimensionless quantity

$$\alpha = \frac{g_5 R}{\sqrt{2\pi R}} v = \sqrt{\frac{NN_5}{\beta\gamma}} \frac{a_4 v}{\pi}, \quad (1.2)$$

with g_5 the five-dimensional coupling, $\pi R = N_5 a_5$ the size of the fifth dimension and we have written this formula for an anisotropic lattice with lattice spacings a_4 and a_5 ($\gamma = a_4/a_5$ in the classical limit). One then finds the preferred value for α by minimizing $V(\alpha)$ [14, 18] and calls it α_{\min} . As a consequence, the otherwise massless gauge boson develops a mass due to this vev equal to

$$m_Z = \frac{\alpha_{\min}}{R}. \quad (1.3)$$

This is the Hosotani mechanism, applied to the case of the orbifold. The result of the analysis of [16], performed at $\gamma = 1$, was that indeed there exist values of $c^{(5)}$ and $c^{(6)}$ that yield a "Mexican hat" Higgs potential that triggers SSB with the Higgs particle having a mass of similar order as the gauge boson. In particular, it was shown that a non-zero $c^{(6)}$ is able to trigger SSB by itself, by shifting α_{\min} from integer (for which there is no SSB)

to half integer. This could be the main phenomenological gain from the complications encountered by entering in the interior of the phase diagram, in view of the fact that in the conventional continuum approaches where one takes $c^{(pi)}(N_5, \beta) = 0$, one necessarily needs fermions in order to trigger SSB [18] (i.e. a non-integer α_{\min}) and even if SSB is achieved, the Higgs typically turns out to be generically too light [19]. In the absence of a non-perturbative control of the theory, in [16] the coefficients $c^{(5,6)}$ were treated as free parameters. Non-perturbatively however they are not free parameters and it is not guaranteed that the quantum theory generates values for the coefficients that trigger SSB.

In this work we improve on these approximations by computing the Wilson loop and the mass spectrum of the lightest states, in the mean-field expansion far from the five-dimensional perturbative point, close to the bulk phase transition. The formalism involved is very similar to the one developed in [1] and therefore will be heavily used. We show that the mean-field expansion predicts the spontaneous breaking of the boundary gauge symmetry already in the pure gauge system and allows for a Higgs like scalar of similar mass as the mass of the broken gauge field. With infinite four-dimensional lattices, the parameters of our model are β , γ and N_5 . To the extent that the mean-field expansion is a good description of the non-perturbative system, any result stemming from this approach should be taken seriously. In fact, the first exploratory Monte Carlo studies of the orbifold theory [12] had earlier reached similar conclusions.

In Section 2 we give a short review of the mean-field expansion formalism. In Section 3 we apply the general formalism to the five-dimensional $SU(2)$ lattice gauge theory with orbifold boundary conditions along the extra dimension. In Section 4 we present our numerical results and in Section 5 our conclusions. In the Appendices we detail the mean-field calculations of the propagator with orbifold boundary conditions and of the mass spectrum.

2 A short review of the mean-field formalism

The partition function of a gauge theory on the lattice is

$$Z = \int DU e^{-S_W[U]}, \quad S_W[U] = \frac{\beta}{2N} \sum_p \text{Re tr}\{1 - U(p)\}, \quad (2.1)$$

where $S_W[U]$ is the Wilson plaquette action and p denotes oriented plaquettes (i.e. each plaquette is counted with two orientations). In the mean-field approach [2] the link variables U are traded for the complex quantities H and V and in terms of these one rewrites the partition function as

$$Z = \int DV \int DH e^{-S_{\text{eff}}[V, H]}, \quad S_{\text{eff}} = S_W[V] + u(H) + (1/N) \text{Re tr}\{HV\}, \quad (2.2)$$

where the effective mean-field action $u(H)$ is defined via

$$e^{-u(H)} = \int DU e^{(1/N)\text{Re tr}\{UH\}}. \quad (2.3)$$

The mean-field or zeroth order approximation amounts to finding the minimum of the effective action when

$$H \longrightarrow \bar{H}\mathbf{1}, \quad V \longrightarrow \bar{V}\mathbf{1}, \quad S_{\text{eff}}[\bar{V}, \bar{H}] = \text{minimal}. \quad (2.4)$$

The zeroth order saddle point solution or "mean-field background" $[\bar{V}]$ can be easily obtained by taking derivatives of Eq. (2.2) with respect to V and H and require them to vanish. One then has

$$\bar{V} = -\frac{\partial u}{\partial H} \Big|_{\bar{H}}, \quad \bar{H} = -\frac{\partial S_W[V]}{\partial V} \Big|_{\bar{V}}. \quad (2.5)$$

The above two equations are the ones that make the action extremal and define the mean-field solution to zeroth order. The free energy per lattice site is

$$F = -\frac{1}{N} \ln(Z). \quad (2.6)$$

At 0'th order we simply have

$$F^{(0)} = \frac{S_{\text{eff}}[\bar{V}, \bar{H}]}{N}. \quad (2.7)$$

Gaussian fluctuations are defined by setting

$$H = \bar{H} + h \quad \text{and} \quad V = \bar{V} + v. \quad (2.8)$$

We impose a covariant gauge fixing on v . In [20] it was shown that this is equivalent to gauge-fix the original links U . The integral

$$z = \int Dv \int Dh e^{-S^{(2)}[v, h]} = \frac{(2\pi)^{|h|/2} (2\pi)^{|v|/2}}{\sqrt{\det[(-\mathbf{1} + K^{(hh)} K^{(vv)})]}} \quad (2.9)$$

introduces the pieces of the propagator

$$\frac{\delta^2 S_{\text{eff}}}{\delta H^2} \Big|_{\bar{V}, \bar{H}} h^2 = h_i K_{ij}^{(hh)} h_j = h^T K^{(hh)} h \quad (2.10)$$

$$\frac{\delta^2 S_{\text{eff}}}{\delta V \delta H} \Big|_{\bar{V}, \bar{H}} v h = v_i K_{ij}^{(vh)} h_j = v^T K^{(vh)} h \quad (2.11)$$

$$\frac{\delta^2 S_{\text{eff}}}{\delta V^2} \Big|_{\bar{V}, \bar{H}} v^2 = v_i K_{ij}^{(vv)} v_j = v^T K^{(vv)} v \quad (2.12)$$

which will be used extensively later. The quadratic part of the effective action is $S^{(2)}[v, h] = \frac{1}{2} (h^T K^{(hh)} h + 2v^T K^{(vh)} h + v^T K^{(vv)} v)$, $|h|$ and $|v|$ denote the dimensionalities of the fluctuation variables h and v .

We would like to compute the expectation value of observables

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U \mathcal{O}[U] e^{-S_w[U]} \quad (2.13)$$

in the mean-field expansion. To first order it is given by the formal expression [1]

$$\langle \mathcal{O} \rangle = \mathcal{O}[\bar{V}] + \frac{1}{2} \text{tr} \left\{ \frac{\delta^2 \mathcal{O}}{\delta V^2} \bigg|_{\bar{V}} K^{-1} \right\}, \quad (2.14)$$

with

$$K = -K^{(vh)} K^{(hh)^{-1}} K^{(vh)} + K^{(vv)} + K^{(\text{gf})} \quad (2.15)$$

and the second derivative of the observable is taken in the mean-field background. $K^{(\text{gf})}$ is the contribution from the gauge fixing term. $K^{(vh)}$ actually turns out to be proportional to the unit matrix and drops out from all expressions. The free energy at this order becomes

$$F^{(1)} = F^{(0)} - \frac{1}{N} \ln(z). \quad (2.16)$$

To extract the mass spectrum, we denote a generic, gauge invariant, time dependent observable as $\mathcal{O}(t)$ and its connected version as $\mathcal{O}^c(t) = \mathcal{O}(t_0 + t) \mathcal{O}(t_0)$. Defining the correlator

$$C(t) = \langle \mathcal{O}^c(t) \rangle - \langle \mathcal{O}(t_0 + t) \rangle \langle \mathcal{O}(t_0) \rangle, \quad (2.17)$$

to first order in the fluctuations the expression reduces to $C(t) = C^{(1)}(t)$ with

$$C^{(1)}(t) = \frac{1}{2} \text{tr} \left\{ \frac{\delta^{(1,1)} \mathcal{O}^c(t)}{\delta^2 V} K^{-1} \right\}, \quad (2.18)$$

where the notation $\delta^{(1,1)}$ means one derivative acting on each of the $\mathcal{O}(t_0 + t)$ and $\mathcal{O}(t_0)$. The mass of the lowest lying state is then

$$m = \lim_{t \rightarrow \infty} \ln \frac{C^{(1)}(t)}{C^{(1)}(t-1)}. \quad (2.19)$$

It turns out that in order to extract the mass of the vector one needs to go to second order in the mean-field expansion. Physical expectation values are formally given at this order by [1]

$$\begin{aligned} \langle \mathcal{O} \rangle &= \mathcal{O}[\bar{V}] + \frac{1}{2} \left(\frac{\delta^2 \mathcal{O}}{\delta V^2} \right)_{ij} (K^{-1})_{ij} \\ &+ \frac{1}{24} \sum_{i,j,l,m} \left(\frac{\delta^4 \mathcal{O}}{\delta V^4} \right)_{ijklm} \left((K^{-1})_{ij} (K^{-1})_{lm} + (K^{-1})_{il} (K^{-1})_{jm} + (K^{-1})_{im} (K^{-1})_{jl} \right). \end{aligned} \quad (2.20)$$

To extract the mass from the connected correlator is straightforward. Again, all time independent contribution (self energies) cancel from connected correlators and at the end the mass is obtained from

$$m = \lim_{t \rightarrow \infty} \ln \frac{C^{(1)}(t) + C^{(2)}(t)}{C^{(1)}(t-1) + C^{(2)}(t-1)} \quad (2.21)$$

where $C^{(2)}(t)$ is the next to leading order correction to the Z -boson correlator.

3 The lattice orbifold in the mean-field expansion

We will now apply the formalism we described in general terms to a specific example: an $SU(2)$ lattice gauge theory in 5 dimensions with Dirichlet boundary conditions for certain components of the gauge field along the fifth dimension.

The discretized version of the S^1/\mathbb{Z}_2 orbifold defined on a five-dimensional Euclidean lattice was constructed in [21]. The points on the lattice are labeled by integer coordinates $n \equiv \{n_M\}$ but we will often use the notation $n_0 = t$ for the time component. The periodic spatial directions ($M = \mu = 1, 2, 3$) have dimensionless extent $L = l/a_4$ and the time-like direction ($M = 0$) has extent T . The fifth dimension ($M = 5$) has extent $N_5 = \pi R/a_5$. The gauge-unfixed anisotropic mean-field Wilson plaquette action reads

$$\begin{aligned} S_{\text{eff}} = & -\frac{\beta_4}{2} \sum_{n_\mu} \sum_{n_5=1}^{N_5-1} \left[\sum_{\mu < \nu} \text{Re tr } V_{p \notin \text{bound}}(n; \mu, \nu) \right] \\ & -\frac{\beta_5}{2} \sum_{n_\mu} \sum_{n_5=0}^{N_5-1} \left[\sum_{\mu} \text{Re tr } V_{p \notin \text{bound}}(n; \mu, 5) \right] \\ & -\frac{\beta_4}{4} \sum_{n_\mu} \left[\sum_{\mu < \nu} \sum_{n_5=0, N_5} \text{Re tr } V_{p \in \text{bound}}(n; \mu, \nu) \right] \\ & + \sum_{n_\mu} \sum_{n_5=1}^{N_5-1} \sum_{\mu} \left[u_2(\rho(n, \mu)) + \sum_{\alpha} h_{\alpha}(n, \mu) v_{\alpha}(n, \mu) \right] \\ & + \sum_{n_\mu} \sum_{n_5=0}^{N_5-1} \left[u_2(\rho(n, 5)) + \sum_{\alpha} h_{\alpha}(n, 5) v_{\alpha}(n, 5) \right] \\ & + \sum_{n_\mu} \sum_{\mu} \sum_{n_5=0, N_5} \left[u_1(\rho(n, \mu)) + \sum_{\alpha} h_{\alpha}(n, \mu) v_{\alpha}(n, \mu) \right], \quad (3.1) \end{aligned}$$

where the effective mean-field actions u_1 and u_2 are computed in Appendix A. The couplings are defined as

$$\beta_4 = \frac{2Na_5}{g_5^2}, \quad \beta_5 = \frac{2Na_4^2}{g_5^2 a_5}. \quad (3.2)$$

In this work we will parametrize the infinite anisotropic lattice by the parameters β and γ , where $\beta_4 = \beta/\gamma$ and $\beta_5 = \beta\gamma$. A gauge transformation acts on a bulk link as

$$U(n, M) \longrightarrow \Omega^{(SU(2))}(n)U(n, M)\Omega^{(SU(2))\dagger}(n + \hat{M}) \quad (3.3)$$

on a boundary link as

$$U(n, M) \longrightarrow \Omega^{(U(1))}(n)U(n, M)\Omega^{(U(1))\dagger}(n + \hat{M}) \quad (3.4)$$

and on a link whose one end is in the bulk and the other touches the boundary as

$$U(n, M) \longrightarrow \Omega^{(U(1))}(n)U(n, M)\Omega^{(SU(2))\dagger}(n + \hat{M}) . \quad (3.5)$$

One possibility is to derive the orbifold theory from its parent circle theory. In this setup a general link satisfies the orbifold projection condition

$$\Gamma U(n, M) = U(n, M), \quad \Gamma = \mathcal{T}_g \mathcal{R} \quad (3.6)$$

where the reflection property about the origin of the fifth dimension is

$$\begin{aligned} \mathcal{R} U(n, \mu) &= U(\bar{n}, \mu) \\ \mathcal{R} U(n, 5) &= U^\dagger(\bar{n} - \hat{5}, 5) \end{aligned} \quad (3.7)$$

with

$$n = (n_\mu, n_5), \quad \bar{n} = (n_\mu, -n_5). \quad (3.8)$$

The transformation property under group conjugation is

$$\mathcal{T}_g U(n, M) = g U(n, M) g^{-1} . \quad (3.9)$$

The boundary conditions that the above projections imply at their fixed points amount to Dirichlet boundary conditions for some of the links at the orbifold boundary hyperplanes. Consequently the gauge group variables at the boundaries are restricted to the subgroup of $SU(N)$ invariant under group conjugation by a constant $SU(N)$ matrix g with the property that g^2 is an element of the center of $SU(N)$. Only gauge transformations that commute with g are still a symmetry at the boundaries and thus there, the orbifold breaks explicitly the gauge group. In general, a bulk group G breaks by g to an equal rank subgroup \mathcal{H} on the two boundaries [4].

For $SU(2)$, which is the gauge group of our focus, we will take $g = -i\sigma^3$. This means that at the orbifold fixed points the gauge group $SU(2)$ is broken to the $U(1)$ subgroup parametrized by $\exp(i\phi\sigma^3)$, where ϕ are compact phases. In the continuum limit this implies that A_μ^3 (the “ Z gauge boson”) and $A_5^{1,2}$ (the “Higgs”) satisfy Neumann boundary conditions and $A_\mu^{1,2}$ and A_5^3 Dirichlet ones.

In the mean-field approach, we parametrize the fluctuating fields in the bulk as

$$\begin{aligned} V(m, M) &= v_0(n, M) + i \sum_{A=1}^3 v_A(n, M) \sigma^A, \\ H(m, M) &= h_0(n, M) - i \sum_{A=1}^3 h_A(n, M) \sigma^A. \end{aligned} \quad (3.10)$$

The σ^A are the Pauli matrices. On the boundaries instead, we use the parameterizations

$$\begin{aligned} V(n, M) &= v_0(n, M) + i v_3(n, M) \sigma^3, \\ H(n, M) &= h_0(n, M) - i h_3(n, M) \sigma^3. \end{aligned} \quad (3.11)$$

with $v_{0,A} \in \mathbb{C}$. For later convenience we define the line

$$l^{(n_5)}(t_0, \vec{m}) = \prod_{m_5=0}^{n_5-1} V((t_0, \vec{m}, m_5); 5) \quad (3.12)$$

and introduce the matrices

$$\sigma^\alpha = \{\mathbf{1}, i\sigma^A\}, \quad \bar{\sigma}^\alpha = \{\mathbf{1}, -i\sigma^A\}, \quad A = 1, 2, 3. \quad (3.13)$$

For the computation of the Z and Higgs masses we first define the orbifold projected Polyakov loop

$$P^{(0)}(t, \vec{m}) = l^{(N_5)}(t, \vec{m}) g l^{(N_5)\dagger}(t, \vec{m}) g^\dagger, \quad (3.14)$$

satisfying $\Gamma P^{(0)} = P^{(0)}$, in terms of which we define the field $\Phi^{(0)}(t, \vec{m}) = P^{(0)}(t, \vec{m}) - P^{(0)\dagger}(t, \vec{m})$ and then the displaced Polyakov loop [1]

$$Z_k^{(0),A}(t, \vec{m}) = \bar{\sigma}^A V((t, \vec{m}, 0); k) \Phi^{(0)\dagger}(t, \vec{m} + \hat{k}) V((t, \vec{m}, 0); k)^\dagger \Phi^{(0)}(t, \vec{m}), \quad (3.15)$$

which assigns a vector and a gauge index to the observable appropriate to a gauge boson. The Higgs observable is derived from the averaged over space and time location connected correlator

$$\mathcal{O}_H^c(t) = \frac{1}{L^6 T} \sum_{t_0} \sum_{\vec{m}', \vec{m}''} \text{tr}\{P^{(0)}(t_0, \vec{m}')\} \text{tr}\{P^{(0)}(t_0 + t, \vec{m}'')\} \quad (3.16)$$

and the Z -boson from the correlator

$$\mathcal{O}_Z^c(t) = \frac{1}{L^6 T} \sum_{t_0} \sum_{\vec{m}', \vec{m}''} \sum_A \sum_k \text{tr}\{Z_k^{(0),A}(t_0, \vec{m}')\} \text{tr}\{Z_k^{(0),A}(t_0 + t, \vec{m}'')\}. \quad (3.17)$$

Out of the above defined objects one can straightforwardly extract the masses using the general results of the previous section.

3.1 The mean-field background

In order to determine the background we need the effective potentials u_1 and u_2 . The effective potentials in the bulk are the same as on the torus, while on the boundaries they are

$$u_1(H(n, M)) = -\ln(I_0(\rho)), \quad \rho = \sqrt{(\text{Re } h_0(n, M))^2 + (\text{Re } h_3(n, M))^2}. \quad (3.18)$$

For details see Appendix A.

Translation invariance along the dimensions $\mu = 0, 1, 2, 3$ means that we can parametrize the saddle point solution, which minimizes S_{eff} , as follows [22]: for $n_5 = 0, 1, \dots, N_5$ (four-dimensional links)

$$\overline{H}(n, \mu) = \overline{h}_0(n_5) \mathbf{1}, \quad \overline{V}(n, \mu) = \overline{v}_0(n_5) \mathbf{1}, \quad \forall n_\mu, \mu, \quad (3.19)$$

and for $n_5 = 0, 1, \dots, N_5 - 1$ (extra-dimensional links)

$$\overline{H}(n, 5) = \overline{h}_0(n_5 + 1/2) \mathbf{1}, \quad \overline{V}(n, 5) = \overline{v}_0(n_5 + 1/2) \mathbf{1}, \quad \forall n_\mu. \quad (3.20)$$

The action at zeroth order reads ($\mathcal{N} = L^3 T \mathcal{N}_5$, $\mathcal{N}_5 = N_5 + 1$)

$$\begin{aligned} \frac{S_{\text{eff}}[\overline{V}, \overline{H}]}{\mathcal{N}} = & \frac{1}{\mathcal{N}_5} \left\{ -\frac{\beta_4}{2} (d-1)(d-2) \left[\sum_{n_5=1}^{N_5-1} \overline{v}_0(n_5)^4 + \frac{1}{2} \overline{v}_0(0)^4 + \frac{1}{2} \overline{v}_0(N_5)^4 \right] \right. \\ & - \beta_5 (d-1) \sum_{n_5=0}^{N_5-1} \overline{v}_0(n_5) (\overline{v}_0(n_5 + 1/2))^2 \overline{v}_0(n_5 + 1) \\ & + (d-1) \left[u_1(\overline{h}_0(0)) + u_1(\overline{h}_0(N_5)) + \sum_{n_5=1}^{N_5-1} u_2(\overline{h}_0(n_5)) + \sum_{n_5=0}^{N_5} \overline{h}_0(n_5) \overline{v}_0(n_5) \right] \\ & \left. + \sum_{n_5=0}^{N_5-1} [u_2(\overline{h}_0(n_5 + 1/2)) + \overline{h}_0(n_5 + 1/2) \overline{v}_0(n_5 + 1/2)] \right\}. \quad (3.21) \end{aligned}$$

The minimization equations lead to the following relations: for $n_5 = 0$

$$\overline{v}_0(0) = -u'_1(\overline{h}_0(0)) = \frac{I_1(\overline{h}_0(0))}{I_0(\overline{h}_0(0))}, \quad (3.22)$$

$$\overline{h}_0(0) = \beta_4 [(d-2)(\overline{v}_0(0))^3 + \gamma^2 (\overline{v}_0(1/2))^2 \overline{v}_0(1)]. \quad (3.23)$$

A prime on u_1 or u_2 denotes differentiation with respect to its argument. Similarly, for $n_5 = N_5$ we have

$$\overline{v}_0(N_5) = -u'_1(\overline{h}_0(N_5)) = \frac{I_1(\overline{h}_0(N_5))}{I_0(\overline{h}_0(N_5))}, \quad (3.24)$$

$$\overline{h}_0(N_5) = \beta_4 [(d-2)(\overline{v}_0(N_5))^3 + \gamma^2 \overline{v}_0(N_5 - 1) (\overline{v}_0(N_5 - 1/2))^2]. \quad (3.25)$$

For $n_5 = 1, \dots, N_5 - 1$ (four-dimensional links)

$$\bar{v}_0(n_5) = -u'_2(\bar{h}_0(n_5)) = \frac{I_2(\bar{h}_0(n_5))}{I_1(\bar{h}_0(n_5))}, \quad (3.26)$$

$$\begin{aligned} \bar{h}_0(n_5) = & \beta_4 \left[2(d-2)(\bar{v}_0(n_5))^3 + \gamma^2 \left((\bar{v}_0(n_5 + 1/2))^2 \bar{v}_0(n_5 + 1) \right. \right. \\ & \left. \left. + \bar{v}_0(n_5 - 1)(\bar{v}_0(n_5 - 1/2))^2 \right) \right]. \end{aligned} \quad (3.27)$$

For $n_5 = 0, \dots, N_5 - 1$ (extra-dimensional links)

$$\bar{v}_0(n_5 + 1/2) = -u'_2(\bar{h}_0(n_5 + 1/2)) = \frac{I_2(\bar{h}_0(n_5 + 1/2))}{I_1(\bar{h}_0(n_5 + 1/2))}, \quad (3.28)$$

$$\bar{h}_0(n_5 + 1/2) = 2\beta_5(d-1)\bar{v}_0(n_5)\bar{v}_0(n_5 + 1/2)\bar{v}_0(n_5 + 1). \quad (3.29)$$

3.2 Observables from fluctuations around the background

In sect. 2 we described the general formalism for computing observables from fluctuations around the mean-field background. Here we apply this formalism to our case and give the results for the free energy, the scalar and vector masses.

The lattice propagator, Fourier transformed along the spatial and time directions is an object that contains the information about the boundary conditions. We denote its components as

$$K^{-1} = K^{-1}(p', n'_5, M', \alpha'; p'', n''_5, M'', \alpha''). \quad (3.30)$$

The momenta p', p'' are four dimensional momenta, however we will further split the momenta into their time and spatial components: $p = p_0, p_k$ whenever it is necessary, for clarity. For a more detailed computation of the propagator we divert the reader at this point to Appendix B.

3.2.1 The free energy

The free energy to first order is

$$F^{(1)} = F^{(0)} + \frac{1}{2\mathcal{N}} \ln \left[\prod_{\alpha=0}^3 \det \left(\Upsilon(p', n'_5, M', \alpha; p'', n''_5, M'', \alpha) D_{\text{FP}}^{(\text{orb})-2} \right) \right], \quad (3.31)$$

where $D_{\text{FP}}^{(\text{orb})}$ is the determinant of the Faddeev-Popov matrix, also computed in Appendix B. The matrix Υ is defined as

$$K^{-1} = \Upsilon^{-1} K^{(hh)}, \quad \Upsilon = -\mathbf{1} + K^{(hh)}(K^{(vv)} + K^{(\text{gf})}). \quad (3.32)$$

There are torons both in the matrix Υ and $D_{\text{FP}}^{(\text{orb})}$, which can be regularized as on the torus [1], see the end of Appendix B.

3.2.2 The Higgs and Z-boson masses

For the Higgs, there is a non-trivial contribution already at first order. The result is

$$C_H^{(1)}(t) = \frac{8}{\mathcal{N}^{(4)}} (P_0^{(0)})^2 \Pi_{(1,1)}^{(1)}(0,0), \quad (3.33)$$

where $P_0^{(0)}$ is the Polyakov loop Eq. (3.14) evaluated on the background and $\Pi_{(1,1)}^{(1)}(0,0)$ is defined in Eq. (C.2). The above correlator does not contain torons since the 00 component of the propagator does not contain any.

For the Z the result is

$$C_Z^{(2)}(t) = \frac{4096}{(\mathcal{N}^{(4)})^2} (P_0^{(0)})^4 (v_0(0))^4 \sum_{\vec{p}'} \sum_k \sin^2 p'_k \Pi_{(1,1)}^{(2)}(1,1)^2, \quad (3.34)$$

where $\Pi_{(1,1)}^{(2)}(1,1)$ is defined in Eq. (C.2). It contains regularizable torons, that is simultaneous zero modes in the propagator and the observable, whose contribution vanish in the infinite lattice volume limit.

Derivations can be found in Appendix C.

3.2.3 The static potential

On the orbifold we have the three types of static potentials. Here we will be interested in the potentials extracted from Wilson loops in the four-dimensional hyperplanes, along either one of the boundaries and in the middle of the orbifold (i.e. at $n_5 = N_5/2$). We consider the Wilson loops of size r along one of the three spatial dimensions and we average over the possible orientations. The exchange contribution (to $\delta^2 \mathcal{O}^c / \delta V^2$) at $n_5 = 0$ is

$$\begin{aligned} \mathcal{O}_{\text{ex}} \equiv & \frac{t^2}{L^3 T} 2(\bar{v}_0(0))^{2(t+n_3)-2} \delta_{M'0} \delta_{M''0} \\ & \delta_{n_5 0} (\delta_{\alpha'0} \delta_{\alpha''0} + \delta_{\alpha'3} \delta_{\alpha''3}) \delta_{p'_0 0} \delta_{p''_0 0} \left(\prod_{M=1,2,3} \delta_{p'_M p''_M} \right) \frac{1}{3} \sum_{k=1}^3 2 \cos(p_k r) \delta_{n'_5 0} \delta_{n''_5 0} \end{aligned} \quad (3.35)$$

and the self energy contributions

$$\begin{aligned} \mathcal{O}_{\text{se}} \equiv & \frac{t^2}{L^3 T} 2(\bar{v}_0(0))^{2(t+n_3)-2} \delta_{M'0} \delta_{M''0} \\ & \delta_{n_5 0} (\delta_{\alpha'0} \delta_{\alpha''0} - \delta_{\alpha'3} \delta_{\alpha''3}) \delta_{p'_0 0} \delta_{p''_0 0} \left(\prod_{M=1,2,3} \delta_{p'_M p''_M} \right) 2 \delta_{n'_5 0} \delta_{n''_5 0}. \end{aligned} \quad (3.36)$$

As for the torus [1], to first order we would like to compute

$$C_W^{(1)} = \frac{1}{2} \sum_{\alpha', \alpha''} \sum_{p'_k} \sum_{n'_5, n''_5} \mathcal{O} \left(0, p'_k, n'_5, 0, \alpha'; 0, p'_k, n''_5, 0, \alpha'' \right) K^{-1} \left(0, p'_k, n'_5, 0, \alpha'; 0, p'_k, n''_5, 0, \alpha'' \right), \quad (3.37)$$

where $\mathcal{O} = \mathcal{O}_{\text{ex}} + \mathcal{O}_{\text{se}}$, which is to be substituted in the general expression for the first order corrected static potential

$$V = \text{const.} - \lim_{t \rightarrow \infty} \frac{1}{t} \frac{C_W^{(1)}}{\mathcal{O}[\bar{V}]}. \quad (3.38)$$

Applied to the gauge boson exchange between two static charges on the boundary the general formula reduces to

$$\begin{aligned} V_4(0) = & -\log(\bar{v}_0(0)^2) - \frac{1}{2} \frac{1}{L^3 T} \frac{1}{(\bar{v}_0(0))^2} \sum_{p'_k} \\ & \left\{ \frac{1}{3} \sum_k \left[2 \cos(p'_k r) + 2 \right] K^{-1}(0, p'_k, 0, 0, 0; 0, p'_k, 0, 0, 0) \right. \\ & \left. + \frac{1}{3} \sum_k \left[2 \cos(p'_k r) - 2 \right] K^{-1}(0, p'_k, 0, 0, 3; 0, p'_k, 0, 0, 3) \right\}. \end{aligned} \quad (3.39)$$

The formula for the static potential $V_4(N_5/2)$ for the static potential along the four dimensional hyperplane in the middle of the orbifold is similar to Eq. (3.39), the difference being that the background and the propagator are evaluated at $n_5 = N_5$ and in the last line of Eq. (3.39) there is a sum over all the gauge components $\alpha = 1, 2, 3$.

4 Spontaneous symmetry breaking

4.1 The phase diagram

Using the equations that determine the mean-field background in Section 3.1, one can extract the leading order approximation to the phase diagram. The equations are solved iteratively and numerically. The confined phase is defined as the phase where $\bar{v}_0(n_5) = 0 = \bar{v}_0(n_5 + 1/2)$ for all n_5 . When $\bar{v}_0(n_5) \neq 0$ and $\bar{v}_0(n_5 + 1/2) = 0$ for all n_5 , we define the layered phase. The Coulomb phase is defined where $\bar{v}_0(n_5) \neq 0$ and $\bar{v}_0(n_5 + 1/2) \neq 0$ for all n_5 . We do not find a phase where $\bar{v}_0(n_5) = 0$ and $\bar{v}_0(n_5 + 1/2) \neq 0$ for all n_5 . The background is sensitive to β, γ and N_5 . On Fig. 1 we plot the phase diagram with color code, red for the confined phase, blue for the layered phase and white for the Coulomb phase. Green is used where for some reason the iterative process does not converge to a solution. For the rest of this paper we will stay in the Coulomb phase.

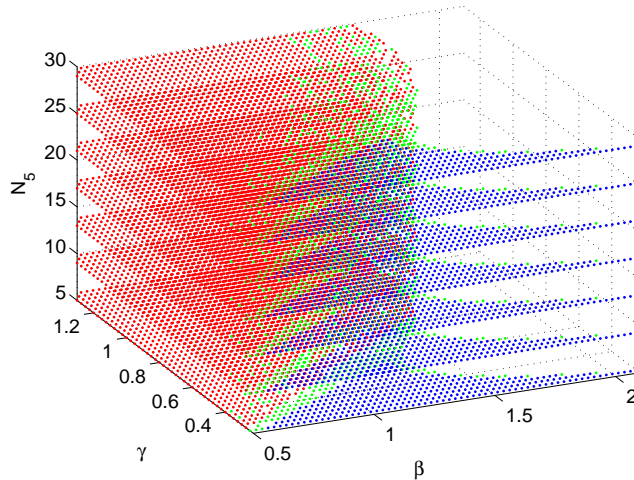


Figure 1: The mean-field phase diagram of the $SU(2)$ orbifold theory in the (β, γ, N_5) space. The color code is explained in the text.

As for the torus, the line that separates the Coulomb from the confined phase is of first order for γ larger than a value which is close to 0.7., below which it turns into a second order phase transition. The order of the phase transition that the mean-field predicts must be taken with care though, as a more careful, fully non-perturbative analysis should be done.

Next, using the quantities $C_W^{(1)}$, $C_Z^{(2)}$ and $C_H^{(1)}$ we analyze the physical properties of the system on this phase diagram, with an emphasis on the issue of spontaneous symmetry breaking (SSB).

4.2 The Higgs

The Higgs mass in units of the lattice spacing $M_H = a_4 m_H$, extracted from $C_H^{(1)}$ in Eq. (3.33), depends on β, γ and N_5 . The physical quantity of our interest is the Higgs mass in units of the radius of the fifth dimension

$$F_1 = m_H R = M_H \frac{N_5}{\gamma \pi}. \quad (4.1)$$

In perturbation theory, the one-loop result [15] for $SU(N)$, expressed in lattice parameters (relevant for the isotropic lattice) is $(C_2(N) = (N^2 - 1)/(2N))$

$$M_H^{\text{pert.}} = \frac{c \gamma \pi}{N_5^{3/2} \beta^{1/2}}, \quad c = \frac{3}{4\pi^2} \sqrt{N \zeta(3) C_2(N)}. \quad (4.2)$$

On the left plot in Fig. 2 we show the N_5 -dependence of M_H for $\gamma = 1$ at $\beta = 1.677$ near the phase transition. We can see clearly that the perturbative formula is not valid at a

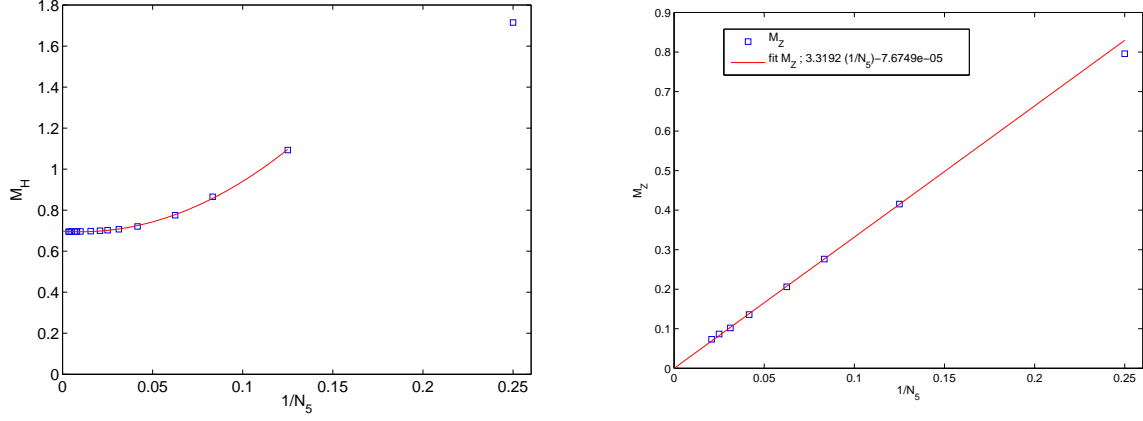


Figure 2: The Higgs mass M_H (left plot) and the Z boson mass M_Z (right plot) as a function of $1/N_5$ at $\gamma = 1$ for $\beta = 1.677$. The squares are mean-field data, the line on the right plot is a quadratic fit and the line on the left plot a linear fit.

generic point on the phase diagram. The line on the left plot in Fig. 2 is a quadratic fit. The phase transition is of first order, which means that the mass in lattice units M_H cannot be lowered to zero but approaches a non-zero minimal value, which at $\beta = 1.677$ is approximately 0.69.

4.3 The Z boson

The Wilson loop can decide if there is SSB. We will choose our lattices so that the system is dimensionally reduced to four dimensions. Then we can describe the boundary gauge theory in four-dimensional terms. If the boundary $U(1)$ symmetry is spontaneously broken then the corresponding static potential extracted from $C_W^{(1)}$ Eq. (3.37) should be fitted by a (4d) Yukawa form rather than by a Coulomb form. Starting from

$$V_4(r) = -b \frac{e^{-m_Z r}}{r} + \text{const.}, \quad (4.3)$$

where b is a constant, we define the quantity $y(r) = \log(r^2 F_4(r))$ where $F_4(r) = dV_4(r)/dr$ from which we form the combination

$$a_4 y'(r) = -M_Z + \frac{M_Z}{m_Z r + 1}. \quad (4.4)$$

By $M_Z = a_4 m_Z$ we denote the Z mass in lattice units. We then determine M_Z iteratively by requiring that a plateau for $-a_4 y'(r) + M_Z/(m_Z r + 1)$ forms. The plateaus, for large enough L , stabilize as L is further increased, so that M_Z at infinite L depends on β , γ and N_5 . Note that in the case of a first order phase transition this corresponds to a Z boson in an infinite physical volume at a finite lattice spacing.

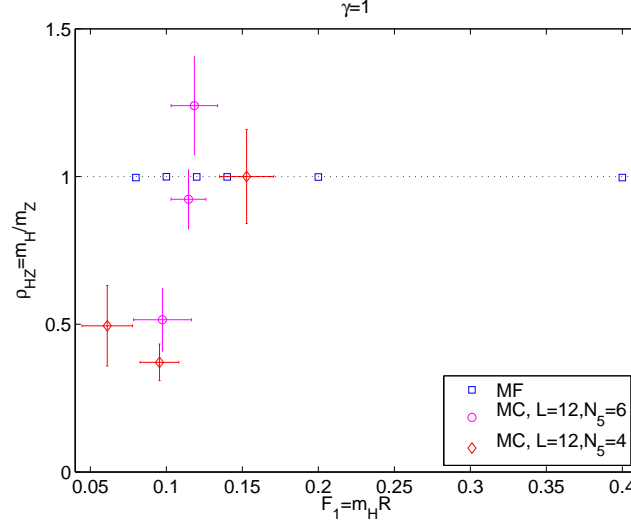


Figure 3: The ratio of the Higgs to the Z boson mass Eq. (4.5). Comparison of Monte Carlo (diamonds [12] and circles [16]) and mean-field data (squares) at $\gamma = 1$.

4.3.1 Isotropic lattices

On the right plot of Fig. 2 we show the M_Z plateau values as a function of $1/N_5$ at fixed $\beta = 1.677$ and $\gamma = 1$, near the bulk phase transition. The plateau values of M_Z do not depend on L for $L \geq 200$ and there is no sign of a plateau for a zero mass. A linear fit with slope 3.32, which is very close to π , describes the data very well. The most striking observation is that the boundary gauge boson is massive, pointing to the dynamical spontaneous breaking of the $U(1)$ symmetry. Clearly, since β and γ are kept fixed, the masses on Fig. 2 correspond to different lattice spacings (the location of the phase transition β_c depends on N_5). Nevertheless, from the Kaluza–Klein description we expect to see an approximate $1/N_5$ dependence. Comparing the data of Fig. 2 to Eq. (1.3) our result indicates a value $\alpha_{\min} = 1$. However the regime of Higgs masses in Fig. 2 corresponds to values of $F_1 = m_H R \gg 1$ which is not the regime where the Coleman–Weinberg calculation is performed and for which $F_1 < 1$. In perturbation theory the minimum at $\alpha_{\min} = 1$ is equivalent to the one at $\alpha_{\min} = 0$ and describes a situation without SSB. Clearly this is not the case for the mean-field data.

On Fig. 3 we plot mean-field data (squares) for the ratio

$$\rho_{HZ} = \frac{m_H}{m_Z}, \quad (4.5)$$

obtained using $N_5 = 4, 6, 8$ and $L = 200$. As far as we have checked the mean-field results in Fig. 3 are independent of N_5 . For $\gamma = 1$ and F_1 in the range $[0.08, 0.4]$ the Higgs and the Z boson are almost degenerate in mass and so $\rho_{HZ} \simeq 1$. As a result, $\alpha_{\min} \simeq F_1$ in this range of F_1 values. Contrary to the data shown in Fig. 3, on Fig. 2, where $\alpha_{\min} = 1$,

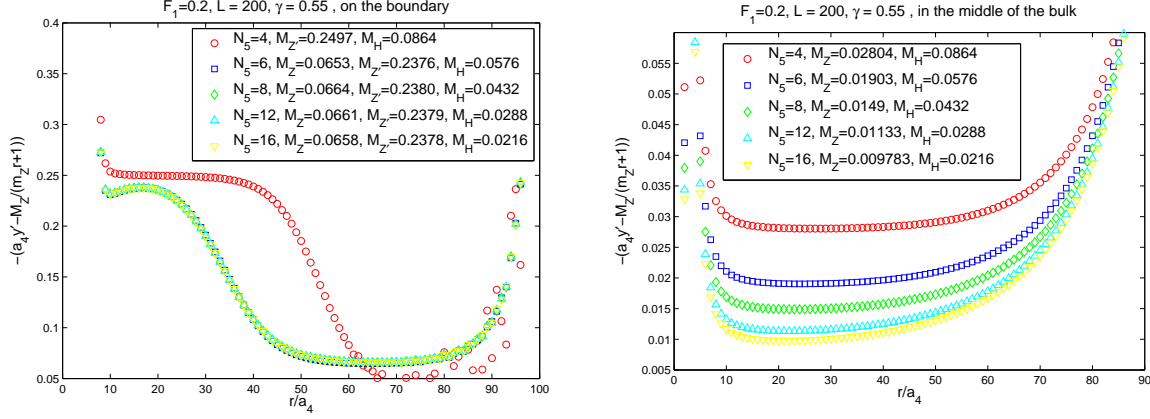


Figure 4: The combination $-a_4 y'(r) + M_Z/(m_Z r + 1)$, cf. Eq. (4.4) is plotted for different values of N_5 at $\gamma = 0.55$ and $F_1 = 0.2$ (for the boundary potential at $N_5 = 4$ we use $M_{Z'}$). Boundary potential (left plot) and bulk potential (right plot).

the values of F_1 are larger than 2 and $\rho_{HZ} > 2$. We compare to the results from Monte Carlo simulations at $N_5 = 4$ (diamonds) [12] and at $N_5 = 6$ (circles) [16], using $L = 12$ and $T = 96$. There is good agreement between the mean-field data and the Monte Carlo data on isotropic lattices, demonstrating that it is possible to obtain values $\rho_{HZ} \geq 1$.

We have also computed the Z boson mass from the potential in the middle of the bulk at $\gamma = 1$. For $F_1 = 0.2$ ($N_5 = 8$), the mass M_Z is the same as the one extracted from the boundary potential. We find $\rho_{HZ} \simeq 1$ in the bulk for $F_1 = 0.2$ and $F_1 = 1.0$.

We find that the mass of the ground state extracted from the direct Z correlator $C_Z^{(2)}$ in Eq. (3.34) at this order in the mean-field expansion does not depend on β or γ and most importantly it does not depend on N_5 . This means that using this observable one can measure only its infinite N_5 limit value, which for finite L , turns out to be

$$M_Z^{\text{dir.}} = \frac{4\pi}{L} \quad (4.6)$$

like on the torus [1]. This expression reflects the fact that this observable describes two non-interacting gluons (that is why the 4π).

4.3.2 Anisotropic lattice

Motivated by Section 4.1 we study SSB at $\gamma = 0.55$ by extracting the Yukawa masses from the static potential on the boundary and in the middle of the orbifold following Eq. (4.4). As we vary N_5 , we set β to keep $F_1 = 0.2$ constant, which means that $M_H \propto 1/N_5$, cf. Eq. (4.1).

On the left plot in Fig. 4 we show the plateaus for the boundary potential. For $N_5 \geq 6$ there are two plateaus corresponding to masses $M_{Z'} > M_Z$ which do not depend on N_5 .

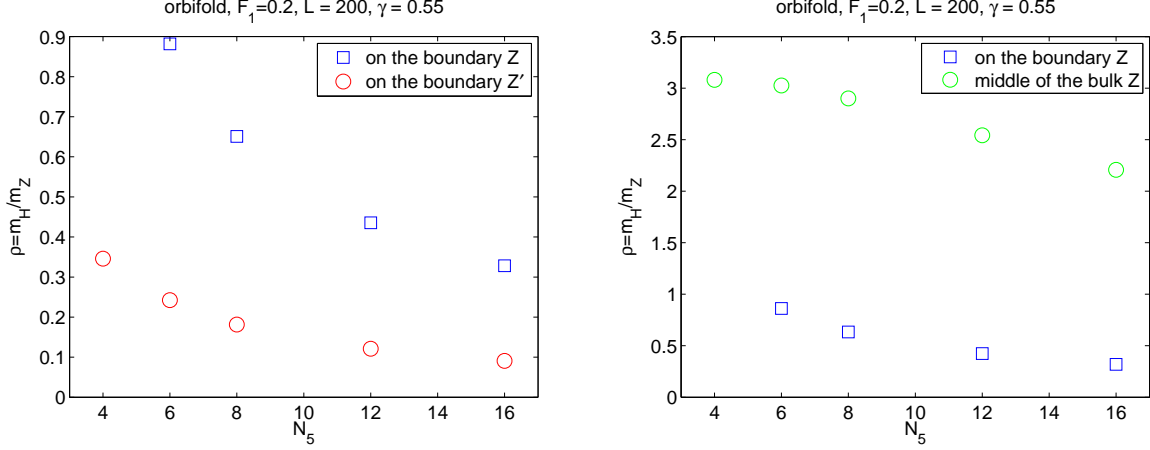


Figure 5: The ratio of the Higgs to the Z boson mass Eq. (4.5) in the mean-field extracted from the static potential. On the boundary (left plot) and in the bulk (right plot).

At $N_5 = 4$ there is only one plateau whose value is very close to $M_{Z'}$ for $N_5 \geq 6$. Therefore we identify the plateau at $N_5 = 4$ with the mass of a Z' boson (although we do not see the Z boson state). We checked that the Yukawa masses are independent of L if L is large. For this we compared $L = 200$ with $L = 300$ and find no differences. These data establish that the boundary theory is a spontaneously broken $U(1)$ theory and by comparing to Eq. (1.3) we get $\alpha_{\min} \approx 0.039 N_5$ for $N_5 \geq 6$. We find $\rho_{HZ} < 1$ for $N_5 \geq 6$ and this particular choice of parameters as shown on the left plot of Fig. 5. Finally we remark that the boundary potential cannot be fitted by a four-dimensional string-like fit.

On the right plot in Fig. 4 we show the situation in the middle of the orbifold. There is one plateau yielding a bulk Z boson mass M_Z . M_Z in the bulk is decreasing as N_5 increases and is independent on L . This result indicates that there is SSB also in the bulk, where we have $\rho_{HZ} > 1$ as shown on the right plot of Fig. 5. The situation on the orbifold is therefore completely different than it is on the torus, where a Yukawa fit yields a mass $\approx 2\pi/L$ implying that there is no SSB. This fact is by itself non trivial as it shows that the orbifold boundary conditions are affecting the properties of the bulk. We observe a difference between the Yukawa masses in the bulk as compared to those on the boundary. This situation is different than the one of the isotropic lattice, where we found the boundary and bulk Yukawa masses to be the same.

5 Conclusions

We have formulated the mean-field expansion in five dimensions on the lattice for an $SU(2)$ gauge theory and orbifold boundary conditions. We have performed computations on the isotropic lattice and for anisotropy parameter $\gamma = 0.55$ and find that the gauge

boson mass extracted from the static potential along four-dimensional hyperplanes is non-zero, both on the boundaries and in the middle of the orbifold. The gauge boson mass does not depend on the spatial size of the lattice, thus indicating dynamical spontaneous symmetry breaking. This result differs from the one obtained in the perturbative limit of this theory, where the gauge boson remains massless, but supports the first Monte Carlo simulations that were performed in [12].

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A Effective mean-field action

The effective mean-field action is defined in Eq. (2.3) as a function of a gauge link $U \equiv U(n, M)$ and a complex matrix $H \equiv H(n, M)$. On the orbifold we have to distinguish two cases, depending whether the link is a bulk $SU(2)$ link or a boundary $U(1)$ link.

A.1 Bulk $SU(2)$ links

In order to compute Eq. (2.3) for gauge group $SU(2)$, we start from the parametrization

$$\begin{aligned} U &= u_0 + i\vec{\sigma} \cdot \vec{u}, \\ H &= h_0 - i\vec{\sigma} \cdot \vec{h}, \end{aligned} \tag{A.1}$$

where u_μ are real and h_μ complex. It is straightforward to compute

$$\begin{aligned} \text{Re tr}\{UH\} &= 2(\text{Re } h_0)u_0 + (\text{Re } h_A)u_A \\ &= \text{Re tr}\{U(\text{Re } h_0 + i\vec{\sigma} \cdot (\text{Re } \vec{h}))\} \end{aligned} \tag{A.2}$$

We can then define an $SU(2)$ matrix $V^\dagger = ((\text{Re } h_0) + i\vec{\sigma} \cdot (\text{Re } \vec{h}))/\rho$ with

$$\rho = \sqrt{(\text{Re } h_0)^2 + (\text{Re } h_A)^2}, \tag{A.3}$$

in terms of which we can write

$$u_2(H) = \int_{SU(2)} DU e^{\frac{1}{2}\text{Re tr}\{UH\}} = \int_{SU(2)} DU e^{\frac{\rho}{2}\text{tr}\{U\}}. \tag{A.4}$$

The integral then is computed using the character expansion of the exponential

$$e^{\frac{\rho}{2}\text{tr}\{U\}} = \frac{2}{\rho} \sum_{\nu} (2\nu + 1) I_{2\nu+1}(\rho) \chi^{(\nu)}(U) \tag{A.5}$$

and the result is

$$u_2(H) = \frac{2}{\rho} I_1(\rho). \quad (\text{A.6})$$

In the mean-field we trade the 3 real degrees of freedom of $SU(2)$ (4 minus 1, from the determinant constraint) for 4 independent mean-field degrees of freedom. The $SU(2)$ memory is encoded in the integral above with the determinant constraint hidden in ρ .

A.2 Boundary $U(1)$ links

The $U(1)$ gauge group of the boundary links is embedded in $SU(2)$ as

$$\begin{aligned} U &= e^{i\phi\sigma^3} = \cos(\phi) + i\sin(\phi)\sigma^3, \\ H &= h_0 - ih_3\sigma^3. \end{aligned} \quad (\text{A.7})$$

The calculation proceeds like in the previous subsection

$$u_1(H) = \int_{U(1) \subset SU(2)} DU e^{\frac{1}{2}\text{Re tr}\{UH\}} = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{\rho \cos(\phi)} = I_0(\rho), \quad (\text{A.8})$$

where

$$\rho = \sqrt{(\text{Re } h_0)^2 + (\text{Re } h_3)^2}. \quad (\text{A.9})$$

B The orbifold propagator

The inverse mean-field propagator on the orbifold is

$$K = -\left(K^{(hh)}\right)^{-1} + \left(K^{(vv)} + K^{(\text{gf})}\right). \quad (\text{B.1})$$

As on the torus, $K^{(vh)}$ drops out from all expressions. The propagator is written as

$$K^{-1} = \Upsilon^{-1} K^{(hh)}, \quad \Upsilon = -\mathbf{1} + K^{(hh)}(K^{(vv)} + K^{(\text{gf})}) \quad (\text{B.2})$$

so that it is the matrix Υ that is inverted when computing an observable.

We will Fourier transform along the four dimensions but not along the fifth dimension. The propagator has then the components

$$K^{-1} = K^{-1}(p', n'_5, M', \alpha'; p'', n''_5, M'', \alpha'') \quad (\text{B.3})$$

and in this Appendix we collect its pieces.

B.1 Fourier transformation to four-dimensional momentum space

The Fourier transformation of a double derivative of an action S

$$\frac{\delta^2 S}{\delta v' \delta v''} \quad (\text{B.4})$$

with respect to $v' \equiv v_{\alpha'}(n', M')$, $v'' \equiv v_{\alpha''}(n'', M'')$ defines a kernel

$$K(p', n'_5, M', \alpha'; p'', n''_5, M'', \alpha''). \quad (\text{B.5})$$

The Fourier transformation is

$$\frac{1}{\mathcal{N}^{(4)}} \sum_{\{n'_\mu\}, \{n''_\mu\}} e^{ip'_\mu n'_\mu} e^{-ip''_\mu n''_\mu} e^{i\frac{p'_M}{2}(1-\delta_{M'5})} e^{-i\frac{p''_M}{2}(1-\delta_{M''5})} (-i)^{\delta_{M'5}} (i)^{\delta_{M''5}} \frac{\delta^2 S}{\delta v' \delta v''}, \quad (\text{B.6})$$

where $\mathcal{N}^{(4)} = TL^3$ is the four-dimensional volume. The factors $(-i)^{\delta_{M'5}}$ and $(i)^{\delta_{M''5}}$ are introduced to make the propagator real, a property that reflects CP invariance [23].

The kernel Eq. (B.5) is a matrix which is divided into sub-matrices in coordinate indices (n'_5, n''_5) for given $(p', M', \alpha'; p'', M'', \alpha'')$. These submatrices have dimension $(N_5 + 1) \times (N_5 + 1)$ if $M' = \mu'$ and $M'' = \mu''$; dimension $N_5 \times (N_5 + 1)$ if $M' = 5$ and $M'' = \mu''$; dimension $N_5 \times N_5$ if $M' = 5$ and $M'' = 5$.

B.2 $K^{(hh)}$ in four-dimensional momentum space

$K^{(hh)}$ on the orbifold is defined as the second derivative

$$\frac{\partial^2}{\partial h_{\alpha'}(n', M') \partial h_{\alpha''}(n'', M'')} \quad (\text{B.7})$$

of the orbifold effective action. We introduce the notation

$$a_2(\alpha, 0) = a_2(\alpha, N_5) = \begin{cases} 0 & \text{if } \alpha = 1, 2 \\ a_2 & \text{if } \alpha = 3 \end{cases} \quad (\text{B.8})$$

and use the quantities

$$a_1 = \frac{1}{2} \left(\frac{u'_1(\rho)}{\rho} + \rho \left(\frac{u'_1(\rho)}{\rho} \right)' \right) \Big|_{\rho=\bar{h}_0(0)=\bar{h}_0(N_5)}, \quad (\text{B.9})$$

$$a_2 = \frac{1}{2} \left(\frac{u'_1(\rho)}{\rho} \right) \Big|_{\rho=\bar{h}_0(0)=\bar{h}_0(N_5)}, \quad (\text{B.10})$$

$$b_1(\cdot) = \frac{u'_2(\rho)}{\rho} + \rho \left(\frac{u'_2(\rho)}{\rho} \right)' \Big|_{\rho=\bar{h}_0(\cdot)}, \quad (\text{B.11})$$

$$b_2(\cdot) = \frac{u'_2(\rho)}{\rho} \Big|_{\rho=\bar{h}_0(\cdot)}. \quad (\text{B.12})$$

As mentioned, we perform a Fourier transformation along only the four-dimensional hyperplanes. The $\alpha' = \alpha'' = 0$ component is

$$K_{00}^{(hh),\text{bulk}} = \delta_{p'p''}^{(4)} \delta_{M'M''} \delta_{n_5'', n_5'} \left\{ (1 - \delta_{M'5}) \sum_{m_5=1}^{N_5-1} \delta_{n_5', m_5} b_1(m_5) + \delta_{M'5} \sum_{m_5=0}^{N_5-1} \delta_{n_5', m_5} b_1(m_5 + 1/2) \right\} \quad (\text{B.13})$$

and

$$K_{00}^{(hh),\text{bound.}} = \delta_{p'p''}^{(4)} \delta_{M'M''} \delta_{n_5'', n_5'} (1 - \delta_{M'5}) (\delta_{n_5', 0} + \delta_{n_5', N_5}) a_1 \quad (\text{B.14})$$

The $\alpha' = \alpha'' = A \neq 0$ components are

$$K_{AA}^{(hh),\text{bulk}} = \delta_{p'p''}^{(4)} \delta_{M'M''} \delta_{n_5'', n_5'} \left\{ (1 - \delta_{M'5}) \sum_{m_5=1}^{N_5-1} \delta_{n_5', m_5} b_2(m_5) + \delta_{M'5} \sum_{m_5=0}^{N_5-1} \delta_{n_5', m_5} b_2(m_5 + 1/2) \right\} \quad (\text{B.15})$$

and

$$K_{AA}^{(hh),\text{bound.}} = \delta_{p'p''}^{(4)} \delta_{M'M''} \delta_{n_5'', n_5'} (1 - \delta_{M'5}) (\delta_{n_5', 0} a_2(A, 0) + \delta_{n_5', N_5} a_2(A, N_5)) \quad (\text{B.16})$$

B.3 $K^{(vv)}$ in four-dimensional momentum space

B.3.1 Gauge fixing

We use the backward derivatives $f^A(n, M) = v^A(n, M) - v^A(n - \hat{M}, M)$, $A = 1, 2, 3$ and introduce the gauge fixing term

$$S_{\text{gf}} = \frac{1}{2\xi} \sum_{\{n_\mu\}} \left\{ \sum_{n_5=1}^{N_5-1} \sum_A \left[\sum_\mu f^A(n, \mu) + \gamma f^A(n, 5) \right]^2 + z_1 \left[\sum_\mu f^3(\{n_\mu\}, n_5 = 0, \mu) \right]^2 + z_1 \left[\sum_\mu f^3(\{n_\mu\}, n_5 = N_5, \mu) \right]^2 \right\}. \quad (\text{B.17})$$

We set the boundary weight to

$$z_1 = \frac{1}{2}. \quad (\text{B.18})$$

The Fourier transformation of $(v' \equiv v_{A'}(n', M'), v'' \equiv v_{A''}(n'', M''))$

$$\frac{\delta^2 S_{\text{gf}}}{\delta v' \delta v''} \quad (\text{B.19})$$

we denote it by

$$K^{(\text{gf})}(p', n_5', M', A'; p'', n_5'', M'', A'') \quad (\text{B.20})$$

and is divided into three contributions.

Contribution 1 is for $M' = \mu'$, $M'' = \mu''$:

$$\begin{aligned} & \frac{1}{\xi} \delta_{p', p''}^{(4)} \hat{p}'_{\mu'} \hat{p}'_{\mu''} \delta_{n'_5, n''_5} \left\{ \delta^{A', A''} \sum_A \delta^{A', A} \sum_{n_5=1}^{N_5-1} \delta_{n'_5, n_5} \right. \\ & \left. + z_1 \delta^{A', A''} \delta^{A', 3} \delta_{n'_5, 0} + z_1 \delta^{A', A''} \delta^{A', 3} \delta_{n'_5, N_5} \right\}, \end{aligned} \quad (\text{B.21})$$

where $\hat{p}_\mu = 2 \sin(p_\mu/2)$.

Contribution 2 is for $M' = 5$, $M'' = \mu''$:

$$\frac{-\gamma}{\xi} \delta_{p', p''}^{(4)} \hat{p}'_{\mu''} \delta^{A', A''} \sum_A \delta^{A', A} \sum_{n_5=1}^{N_5-1} [\delta_{n'_5, n_5} \delta_{n''_5, n'_5} - \delta_{n'_5, n_5-1} \delta_{n''_5, n'_5+1}] . \quad (\text{B.22})$$

The contribution for $M' = \mu'$, $M'' = 5$ is obtained using the Hermiticity property.

Contribution 3 is for $M' = 5$, $M'' = 5$:

$$\frac{\gamma^2}{\xi} \delta_{p', p''}^{(4)} \delta^{A', A''} \sum_A \delta^{A', A} \sum_{n_5=1}^{N_5-1} \square_{n'_5, n''_5}(n_5), \quad (\text{B.23})$$

where

$$\square_{n'_5, n''_5}(n_5) = \delta_{n'_5, n''_5} (\delta_{n'_5, n_5} + \delta_{n'_5, n_5-1}) - \delta_{n''_5, n'_5+1} \delta_{n'_5, n_5-1} - \delta_{n''_5, n'_5-1} \delta_{n'_5, n_5}. \quad (\text{B.24})$$

We have also implemented a background gauge fixing

$$\begin{aligned} f^A(n, \mu) &= \bar{v}_0(n_5) [v^A(n, \mu) - v^A(n - \hat{\mu}, \mu)] , \\ f^A(n, 5) &= \bar{v}_0(n_5 - \frac{1}{2}) v^A(n, 5) - \bar{v}_0(n_5 + \frac{1}{2}) v^A(n - \hat{5}, 5), \end{aligned} \quad (\text{B.25})$$

which is equally valid and vanishes when evaluated on the background \bar{v}_0 . It does not change any of the results for physical observables.

B.3.2 The Faddeev-Popov determinant

Even though not directly relevant for the gauge propagator we can now carry out the Faddeev-Popov construction, necessary for the free energy. The ghost action is

$$S_{\text{FP}} = \sum_{A', A''} \sum_{n', n''} \bar{c}^{A'}(n') \mathcal{M}_{A'n'; A''n''}^{(\text{orb})} c^{A''}(n'') \quad (\text{B.26})$$

where the sums run in the fundamental domain of the orbifold. The ghost kernel

$$\mathcal{M}_{A'n'; A''n''}^{(\text{orb})} = \sum_M \mathcal{M}_{A'n'; A''n''}^{(M)} \quad (\text{B.27})$$

is obtained from the variation under infinitesimal gauge transformations

$$\delta f^{A'}(n, M) = \sum_{A'', n''} \mathcal{M}_{A'n; A''n''}^{(M)} \omega^{A''}(n''). \quad (\text{B.28})$$

The changes on the orbifold with respect to the torus case in [1] is that on the boundaries the gauge transformation is $U(1)$ and acts only on v^3 at the boundaries at $n_5 = 0, N_5$.

We define the matrices

$$\delta^{A', A''} \bar{v}_0(n'_5) (2\delta_{n'', n'} - \delta_{n'', n'+\hat{\nu}} - \delta_{n'', n'-\hat{\nu}}) \quad (\text{B.29})$$

with four-dimensional Fourier transformation

$$\mathcal{M}_{A'n'_5; A''n''_5}^{(\nu)} = \delta^{A', A''} \bar{v}_0(n'_5) \delta_{p', p''}^{(4)} 4 \hat{p}_\nu^2 \delta_{n'_5, n''_5} \quad (\text{B.30})$$

and

$$\begin{aligned} \mathcal{M}_{A'n'_5; A''n''_5}^{(5)} &= \delta_{n''_\nu, n'_\nu} \left[\delta^{A', A''} \left(\bar{v}_0(n'_5 + \frac{1}{2}) + \bar{v}_0(n'_5 - \frac{1}{2}) \right) \delta_{n''_5, n'_5} \right. \\ &\quad - \left[\delta^{A', A''} + \delta_{n'_5, N_5-1} \delta^{A', A''} (\delta^{A', 3} - 1) \right] \bar{v}_0(n'_5 + \frac{1}{2}) \delta_{n''_5, n'_5 + \hat{5}} \\ &\quad \left. - \delta^{A', A''} \left[1 + \delta_{n'_5, 1} (\delta^{A', 3} - 1) \right] \bar{v}_0(n'_5 - \frac{1}{2}) \delta_{n''_5, n'_5 - \hat{5}} \right] \end{aligned} \quad (\text{B.31})$$

with the four-dimensional Fourier transformation changing $\delta_{n''_\nu, n'_\nu} \rightarrow \delta_{p', p''}^{(4)}$. The final expression for the Faddeev-Popov kernel is

$$\begin{aligned} \mathcal{M}_{A'n'_5; A''n''_5}^{(\text{orb})} &= \sum_{l_5=1}^{N_5-1} \delta_{n'_5, l_5} (1 - \delta_{l_5, 0}) (1 - \delta_{l_5, N_5}) \sum_{\nu} \left[\mathcal{M}_{A', n'_5; A'', n''_5}^{(\nu)} + \gamma \mathcal{M}_{A', n'_5; A'', n''_5}^{(5)} \right] \\ &\quad + z_1 \sum_{\nu} \left[\delta_{n'_5, 0} \mathcal{M}_{3, n'_5; 3, n''_5}^{(\nu)} + \delta_{n'_5, N_5} \mathcal{M}_{3, n'_5; 3, n''_5}^{(\nu)} \right]. \end{aligned} \quad (\text{B.32})$$

The relevant for the free energy Faddeev-Popov determinant is

$$D_{\text{FP}}^{(\text{orb})} = \det \mathcal{M}^{(\text{orb})}. \quad (\text{B.33})$$

B.3.3 The double derivative of the plaquette action

As for $K^{(hh)}$ due to the gauge invariance of the Wilson plaquette action the contributions of double derivatives with respect to links to the orbifold propagator simplifies to

$$\begin{aligned} K^{(vv)} &= \frac{\partial^2}{\partial v_{\alpha'}(n', M') \partial v_{\alpha''}(n'', M'')} \left[\sum_{n \in \mathcal{F}_0} \sum_M S_W[U(n, M)] \right] \\ &\quad + \text{boundary terms} \end{aligned} \quad (\text{B.34})$$

with \mathcal{F}_0 the fundamental domain of the orbifold with the boundary contributions separated out.

The gauge structure of the full kernel is

$$K_{\alpha'\alpha''}^{(vv)} = \begin{pmatrix} (0,0) & 0 \\ 0 & (A', A'') \end{pmatrix}, \quad (\text{B.35})$$

where we use the notation (A', A'') for the blocks along the algebra indices. Since

$$\frac{\delta^2 S_W}{\delta v' \delta v''} = \frac{\delta^2 S_W}{\delta v'' \delta v'}, \quad (\text{B.36})$$

the kernel $K^{(vv)}$ is Hermitian. Hence, the off-diagonal elements are related by the Hermiticity relations

$$\left(K_{\alpha'\alpha''}^{(vv)}\right)_{M'M''} = \left(\left(K_{\alpha''\alpha'}^{(vv)}\right)_{M''M'}\right)^\dagger \quad (\text{B.37})$$

where $\left(K_{\alpha'\alpha''}^{(vv)}\right)_{M'M''}$ is a $(N_5 + 1)$ -dimensional matrix.

$K^{(vv)}$ depends on the following weights

$$\begin{aligned} v_1^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} \bar{v}_0(n_5 + 1/2) \bar{v}_0(n_5 + 1), \\ v_2^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} \bar{v}_0(n_5) \bar{v}_0(n_5 + 1/2), \\ u_1^{(\alpha', \alpha'')}(n_5) &= v_1^{(\alpha', \alpha'')}(n_5), \\ u_2^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} \bar{v}_0(n_5 - 1/2) \bar{v}_0(n_5 - 1). \end{aligned} \quad (\text{B.38})$$

and

$$\begin{aligned} w_1^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} (\bar{v}_0(n_5))^2, \\ w_2^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} \bar{v}_0(n_5) \bar{v}_0(n_5 + 1), \\ w_3^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} (\bar{v}_0(n_5 + 1/2))^2, \\ w_4^{(\alpha', \alpha'')}(n_5) &= \delta_{\alpha', \alpha''} (\bar{v}_0(n_5 - 1/2))^2. \end{aligned} \quad (\text{B.39})$$

The background is defined only for $\bar{v}_0(n_5)$, $n_5 = 0, 1/2, 1, 3/2, \dots, N_5$. Outside this range it vanishes by definition. Correspondingly, the above definitions of the weights hold for every value of $n_5 = 0, \dots, N_5$ in the range where the background is defined and for all values of (α', α'') except from a few special cases which are related to certain contributions from boundary or bulk/boundary plaquettes. We specify these special cases below.

To begin, we set for $A' = A'' = 1$ and $A' = A'' = 2$ the weights that originate by

taking at least one derivative on one of the boundaries to zero:

$$\begin{aligned}
w_1^{(1,1)}(0) &= w_1^{(1,1)}(N_5) = 0, \\
w_4^{(1,1)}(N_5) &= 0, \\
v_1^{(1,1)}(0) &= 0, \\
v_2^{(1,1)}(N_5 - 1) &= 0, \\
u_1^{(1,1)}(0) &= u_2^{(1,1)}(N_5) = 0, \\
w_4^{(1,1)}(1) &= w_3^{(1,1)}(N_5 - 1) = 0, \\
w_3^{(1,1)}(0) &= 0.
\end{aligned} \tag{B.40}$$

For the elements $A' = A'' = 3$ we have

$$\begin{aligned}
w_1^{(3,3)}(0) &= 1/2 \bar{v}_0(0)^2, \\
w_1^{(3,3)}(N_5) &= \frac{1}{2} \bar{v}_0(N_5)^2, \\
w_4^{(3,3)}(N_5) &= \bar{v}_0(N_5 - 1/2)^2, \\
u_2^{(3,3)}(N_5) &= \bar{v}_0(N_5 - 1/2) \bar{v}_0(N_5 - 1), \\
w_3^{(3,3)}(N_5 - 1) &= \bar{v}_0(N_5 - 1/2)^2, \\
v_2^{(3,3)}(N_5 - 1) &= \bar{v}_0(N_5 - 1) \bar{v}_0(N_5 - 1/2).
\end{aligned} \tag{B.41}$$

As mentioned, all other weights have their usual value, defined in Eq. (B.38) and Eq. (B.39). Weights that are not defined are assumed to be identically zero. Finally, the anisotropy can be also absorbed in a redefinition of the weights:

$$\begin{aligned}
v_{1,2}^{(\alpha', \alpha'')} &\rightarrow v_{1,2} \gamma, \\
u_{1,2}^{(\alpha', \alpha'')} &\rightarrow u_{1,2} \gamma, \\
w_1^{(\alpha', \alpha'')} &\rightarrow w_1 / \gamma, \\
w_{2,3,4}^{(\alpha', \alpha'')} &\rightarrow w_{2,3,4} \gamma
\end{aligned} \tag{B.42}$$

for every (α', α'') .

We will now compute the various individual contributions to the full kernel. We define the symbols

$$\epsilon(\alpha'; M', M'') = \text{sgn} [\delta_{M'M''} + (1 - \delta_{M'M''})(2\delta_{\alpha',0} - 1)] \tag{B.43}$$

and

$$\begin{aligned}
&e_{(p', n'_5, M', \alpha'; p'', n''_5, M'', \alpha'')}(N', N''; a, b, c, d) = \\
&\delta_{p', p''}^{(4)} e^{i \frac{p'_M}{2} (1 - \delta_{M'5})} e^{-i \frac{p''_M}{2} (1 - \delta_{M''5})} (-i)^{\delta_{M'5}} (i)^{\delta_{M''5}} \\
&\left(a \delta_{n''_5, n'_5} + \epsilon(\alpha'; M', M'') b \delta_{n''_5, n'_5 + \delta_{N'5}} e^{-i p'_{N'} (1 - \delta_{N'5})} \right. \\
&\left. + \epsilon(\alpha'; M', M'') c \delta_{n''_5, n'_5 - \delta_{N''5}} e^{i p'_{N''} (1 - \delta_{N''5})} + d \delta_{n''_5, n'_5 + \delta_{N'5} - \delta_{N''5}} e^{-i p'_{N'} (1 - \delta_{N'5})} e^{i p'_{N''} (1 - \delta_{N''5})} \right).
\end{aligned} \tag{B.44}$$

The double derivatives contribute then the components

$$\begin{aligned}
K^{(vv)}(x'; x'') = & -\beta \cdot \left[\delta_{M'M''} \delta_{M'5} \sum_{\mu} e_{(x'; x'')}(\mu, \mu; 0, w_2, w_2, 0) \right. \\
& + \delta_{M'M''} (1 - \delta_{M'5}) \left(\sum_{\mu \neq M'} e_{(x'; x'')}(\mu, \mu; 0, w_1, w_1, 0) + e_{(x'; x'')}(5, 5; 0, w_3, w_4, 0) \right) \\
& + (1 - \delta_{M'M''}) (1 - \delta_{M'5}) (1 - \delta_{M''5}) e_{(x'; x'')}(M', M''; w_1, w_1, w_1, w_1) \\
& + \delta_{M'5} (1 - \delta_{M''5}) e_{(x'; x'')}(5, M''; v_1, v_2, v_1, v_2) \\
& \left. + (1 - \delta_{M'5}) \delta_{M''5} e_{(x'; x'')}(M', 5; u_1, u_1, u_2, u_2) \right]. \tag{B.45}
\end{aligned}$$

The argument of all weights in the above is n'_5 and their superscript (α', α'') . Also we have defined the collective index $(x'; x'') = (p', n'_5, M', \alpha'; p'', n''_5, M'', \alpha'')$.

To these components, the corresponding components of the contributions from the gauge fixing term must be added, so that the full contribution from the pure gauge part of the action to the propagator is

$$K^{(vv)} + K^{(\text{gf})}. \tag{B.46}$$

We now have all the ingredients to compute K^{-1} in Eq. (B.2). The last issue to be discussed before one does so is its eigenvalue structure.

A general property of Υ is that its eigenvalues are invariant under $p_\mu \rightarrow -p_\mu$ for any $\mu = 0, 1, 2, 3$. This is useful when computing the free energy numerically. Zero eigenvalues in $K^{(hh)}$, if any, clearly do not contribute anything to the free energy or to any of the other observables. The matrix Υ may have certain zero eigenvalues which on the other hand must be taken care of otherwise it cannot be inverted. We separate these zero modes in two classes.

One obtains spurious zero eigenvalues due to the Dirichlet boundary conditions that force some of the fields to vanish on the boundaries. For example, for $\alpha' = \alpha'' = 1$ and $\alpha' = \alpha'' = 2$ the link variables $v'_\alpha(0, \mu)$, $v'_\alpha(N_5, \mu)$ all vanish resulting into $2(d-1)$ ($d=5$) zero eigenvalues in the 11 and 22 components of Υ . These spurious eigenvalues occur for every value of the four-dimensional momenta and appear just because bulk and boundary components are packed together into the propagator. They are unphysical and can be removed by hand.

The second class of zero eigenvalues of Υ contains those corresponding to vanishing four-dimensional momentum $p = 0$. On a finite lattice with fully periodic boundary condition they appear in any gauge invariant formulation and persist even after fixing completely the gauge, due to a left over global gauge invariance surviving on a lattice [2]. Beyond the spurious eigenvalues mentioned above, the following properties hold for any

$\alpha \neq 0$ if $p_\mu = 0 \forall \mu$: without gauge fixing Υ has N_5 zero eigenvalues, corresponding to local gauge transformations of the links along the extra dimension. After gauge fixing through Eq. (B.17) (also in the variant of Eq. (B.25)), only one zero mode survives. This is called “toron”. If $p \neq 0$ then there are $N_5 - 1$ zero eigenvalue which are completely removed by the gauge fixing. When present, such toron zero modes render the corresponding physical observable, plagued by infinities, unusable. Since however it is a finite volume effect, if (and only if) one can ensure the existence of some regularization which leaves behind a finite result, they can be dropped; their contribution to any physical observable is volume suppressed and disappears when the infinite volume is taken [1].

We finally report on the zero eigenvalues of the Faddeev-Popov determinant. When the four dimensional momentum p vanishes there are 2 zero eigenvalues for any α . They correspond to lines of zeros in the Faddeev-Popov matrix \mathcal{M} . If p is not zero there are two zero eigenvalues for $\alpha = 1, 2$ and none for $\alpha = 3$.

C Z and Higgs to leading order

We first define quantities that will be heavily used in the calculation of the observables. Let

$$\Delta_1^{(N_5)}(n_5) = \sum_{r=0}^{N_5-1} \frac{\delta_{n_5, r}}{\bar{v}_0(r + 1/2)} = (1 - \delta_{n_5, N_5}) \frac{1}{\bar{v}_0(n_5 + \frac{1}{2})} \quad (\text{C.1})$$

and

$$\begin{aligned} \Pi_{(1,1)}^{(1)}(\alpha, \beta) &= 2 \sum_{p'_0} \cos p'_0 t \sum_{n'_5, n''_5} \Delta_1^{(N_5)}(n'_5) K^{(-1)}(p'_0, \vec{0}, n'_5, 5, \alpha; p'_0, \vec{0}, n''_5, 5, \beta) \Delta_1^{(N_5)}(n''_5) \\ \Pi_{(1,1)}^{(2)}(\alpha, \beta) &= \sum_{p'_0} e^{ip'_0 t} \sum_{n'_5, n''_5} \Delta_1^{(N_5)}(n'_5) K^{(-1)}(p'_0, \vec{p}', n'_5, 5, \alpha; p'_0, \vec{p}', n''_5, 5, \beta) \Delta_1^{(N_5)}(n''_5). \end{aligned} \quad (\text{C.2})$$

We start from the Z mass correlator. The computation is similar to the one for the torus, so for more details see [1]. First we look at the single derivative of $P^{(0)} - P^{(0)\dagger}$ which after the derivatives leaves behind the matrices

$$\Sigma^\alpha = \sigma^\alpha g g^\dagger + g \bar{\sigma}^\alpha g^\dagger - \text{h.c.}, \quad (\text{C.3})$$

in the background. A simple calculation gives

$$\Sigma^0 = 0, \quad \Sigma^1 = 4i\sigma^1, \quad \Sigma^2 = 4i\sigma^2, \quad \Sigma^3 = 0. \quad (\text{C.4})$$

The group structure contributes terms of the form $\text{tr}\{\bar{\sigma}^3 \Sigma^{\alpha_i \dagger} \Sigma^{\alpha_j}\}$, to be contracted against the Euclidean structure, which we show below:

$$\frac{1}{L^6} \delta_{\vec{p}', \vec{p}'} \delta_{\vec{q}', \vec{q}'} \delta_{\vec{p}', -\vec{q}'} \quad (\text{C.5})$$

times

$$\begin{aligned}
& e^{i(p'_l - p'_k)} \delta^{\alpha_1 A_1} \delta^{\alpha_2 A_2} \delta^{\alpha_3 A_3} \delta^{\alpha_4 A_4} K^{-1}(\alpha_1, \alpha_3) K^{-1}(\alpha_2, \alpha_4) \\
& + e^{i(p'_l + p'_k)} \delta^{\alpha_1 A_2} \delta^{\alpha_2 A_1} \delta^{\alpha_3 A_3} \delta^{\alpha_4 A_4} K^{-1}(\alpha_1, \alpha_3) K^{-1}(\alpha_2, \alpha_4) \\
& + e^{-i(p'_l + p'_k)} \delta^{\alpha_1 A_1} \delta^{\alpha_2 A_2} \delta^{\alpha_3 A_3} \delta^{\alpha_4 A_4} K^{-1}(\alpha_1, \alpha_3) K^{-1}(\alpha_2, \alpha_4) \\
& + e^{-i(p'_l - p'_k)} \delta^{\alpha_1 A_2} \delta^{\alpha_2 A_1} \delta^{\alpha_3 A_4} \delta^{\alpha_4 A_3} K^{-1}(\alpha_1, \alpha_3) K^{-1}(\alpha_2, \alpha_4) \\
& + e^{i(p'_l - p'_k)} \delta^{\alpha_1 A_1} \delta^{\alpha_2 A_2} \delta^{\alpha_3 A_4} \delta^{\alpha_4 A_3} K^{-1}(\alpha_1, \alpha_4) K^{-1}(\alpha_2, \alpha_3) \\
& + e^{i(p'_l + p'_k)} \delta^{\alpha_1 A_2} \delta^{\alpha_2 A_1} \delta^{\alpha_3 A_4} \delta^{\alpha_4 A_3} K^{-1}(\alpha_1, \alpha_4) K^{-1}(\alpha_2, \alpha_3) \\
& + e^{-i(p'_l + p'_k)} \delta^{\alpha_1 A_1} \delta^{\alpha_2 A_2} \delta^{\alpha_3 A_3} \delta^{\alpha_4 A_4} K^{-1}(\alpha_1, \alpha_4) K^{-1}(\alpha_2, \alpha_3) \\
& + e^{-i(p'_l - p'_k)} \delta^{\alpha_1 A_2} \delta^{\alpha_2 A_1} \delta^{\alpha_3 A_3} \delta^{\alpha_4 A_4} K^{-1}(\alpha_1, \alpha_4) K^{-1}(\alpha_2, \alpha_3). \tag{C.6}
\end{aligned}$$

Putting everything together, the observable can be first written as

$$\begin{aligned}
C_Z^{(2)}(t) &= \frac{c^{(2)}}{4!} (P_0^{(0)})^4 (v_0(0))^4 \frac{1}{(\mathcal{N}^{(4)})^2} \sum_{A_1, \dots, A_4} \text{tr}\{\bar{\sigma}^3 \Sigma^{A_1 \dagger} \Sigma^{A_2}\} \text{tr}\{\bar{\sigma}^3 \Sigma^{A_3 \dagger} \Sigma^{A_4}\} \cdot 2 \cdot \sum_{\vec{p}'} \sum_{k,l} \left\{ \right. \\
& e^{i(p'_l - p'_k)} \Pi_{(1,1)}^{(2)}(A_1, A_3) \Pi_{(1,1)}^{(2)}(A_2, A_4) + e^{-i(p'_l - p'_k)} \Pi_{(1,1)}^{(2)}(A_2, A_4) \Pi_{(1,1)}^{(2)}(A_1, A_3) \\
& + e^{i(p'_l + p'_k)} \Pi_{(1,1)}^{(2)}(A_2, A_3) \Pi_{(1,1)}^{(2)}(A_1, A_4) + e^{-i(p'_l + p'_k)} \Pi_{(1,1)}^{(2)}(A_1, A_4) \Pi_{(1,1)}^{(2)}(A_2, A_3) \left. \right\} \tag{C.7}
\end{aligned}$$

and finally be simplified to

$$C_Z^{(2)}(t) = \frac{c^{(2)} \cdot 4 \cdot 4096}{24(\mathcal{N}^{(4)})^2} (P_0^{(0)})^4 (v_0(0))^4 \sum_{\vec{p}'} \sum_k \sin^2 p'_k \cdot \Pi_{(1,1)}^{(2)}(1, 1) \Pi_{(1,1)}^{(2)}(2, 2). \tag{C.8}$$

We denote by $P_0^{(0)}$ the Polyakov loop evaluated on the background. In the above we have used that the elements of the propagator which are diagonal in the M index are invariant under $\vec{p}' \rightarrow -\vec{p}'$. The coefficient $c^{(2)}$ is the symmetry factor of the double exchange diagram and it is

$$c^{(2)} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}. \tag{C.9}$$

We note that the toron ($p_\mu = 0 \forall \mu$) does not contribute to Eq. (C.8).

Next we turn to the Higgs mass. The relevant trace is now

$$\langle 1 \rangle_\alpha = \text{tr}\{\sigma^\alpha g g^\dagger + g \bar{\sigma}^\alpha g^\dagger\}. \tag{C.10}$$

Apart from the traces originating from the above structure, the computation is again analogous to the one for the torus geometry (see [1]) so we give directly the result:

$$C_H^{(1)}(t) = \frac{1}{\mathcal{N}^{(4)}} (P_0^{(0)})^2 \frac{1}{2!} c^{(1)} \sum_\alpha \langle 1 \rangle_\alpha^2 \Pi_{(1,1)}^{(1)}(\alpha, \alpha), \tag{C.11}$$

summation over repeated gauge indices implied. The symmetry factor is

$$c^{(1)} = 1. \quad (\text{C.12})$$

Performing the traces one obtains

$$C_H^{(1)}(t) = \frac{1}{\mathcal{N}^{(4)}} (P_0^{(0)})^2 \frac{16}{2!} \Pi_{\langle 1,1 \rangle}^{(1)}(0,0). \quad (\text{C.13})$$

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